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Three-dimensional solutions for the stress fields in functionally graded cylindrical panel with finite length and subjected to thermal/mechanical loads

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Abstract

Three-dimensional thermo-elastic analysis of a functionally graded cylindrical panel with finite length and subjected to nonuniform mechanical and steady-state thermal loads are carried out in this paper. Thermal and mechanical properties of the functionally graded material are assumed to be temperature independent and continuously vary in the radial direction of the panel. Analytical solutions for the temperature and stress fields expressed in terms of trigonometric and power series for the simply supported boundary conditions are derived and graphically presented.

Keywords: Functionally graded material; Cylindrical panel; Thermal/mechanical stress

1. Introduction

It is well known that many structural components, such as hollow cylinder, thin-walled shell, pipe, cylindrical panel, etc. undergo mechanical and/or thermal loads which may induce undesirable stresses and deformation. How to reduce the aforementioned stresses and deformation becomes important for engineering applications, and extensive effects have been devoted to this field (Tanigawa, 1995). Functionally graded materials (FGM) as a new kind of composites were initially designed as thermal barrier materials for

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aerospace structures, in which the volume fractions of different constituents of composite material vary continuously from one side to another (Suresh and Mortensen, 1998). These novel nonhomogeneous materials have excellent thermo-mechanical properties to withstanding high temperature and have extensive applications to important structures, such as aerospace, nuclear reactors and chemical plants, etc. The use of functionally graded materials can eliminate and/or control thermal deformation of structural components (Aboudi et al., 1994; Wetherhold and Wang, 1996).

Due to the nonhomogeneity of such a kind of novel composite materials and the mathematical difficulty, it is difficult to obtain the analytical thermo-elastic solutions for the stress and temperature fields in functionally graded structures. A widely used method is the so-called multi-layered method, in which each layer are assumed to be homogeneous and continuous conditions between each layers are used to derive the final solutions of the problem. Using the multi-layered method, Ootao and Tanigawa (1999, 2000) studied the three-dimensional transient thermal stresses of functionally graded rectangular plates induced by partial heating, and the three-dimensional transient piezo-thermo-elasticity of functionally graded rectangular plate bonded to a piezoelectric plate. Two-dimensional unsteady thermo-elastic problems of functionally graded infinite hollow cylinder and the deflection of functionally graded plate under transient thermal loading are studied by Kim and Noda (2002a,b) on the basis of the multi-layered method and Green's function approach. Using the multi-layered method and through a novel limiting process, Liew et al. (2003) derived the analytical solutions of thermal stress in functionally graded circular hollow cylinder in terms of the solutions of homogeneous circular hollow cylinder. The multi-layered method is also employed by Shao (2005) to derive the two-dimensional analytical solutions of thermal/mechanical stresses in functionally graded circular hollow cylinder with finite length.

On the other hand, perturbation method was employed by Obata and Noda (1994) to study the one-dimensional steady-state thermal stresses in a functionally graded circular hollow cylinder and hollow sphere, in which the effect of porosity on material properties is considered. Using finite element method, Reddy and Chin (1998) studied the dynamic response of functionally graded cylinders and plates, in which effect of thermo-mechanical coupling on the temperature and stress fields were considered for different thermal loading conditions. Using finite difference method, Awaji and Sivakuman (2001) studied the one-dimensional transient thermo-elasticity of functionally graded circular hollow cylinder.

Additionally, considering the simple models of material properties for FGM, one can obtain the analytical solutions for the mechanical and/or thermo-mechanical problems of FGM structures. Jabbari et al. (2002, 2003) assumed the material properties of FGM obey a power law of distribution of the volume fraction of the constituents and obtained the analytical solutions of one- and two-dimensional steady-state thermo-elastic stresses in functionally graded circular hollow cylinder. Using the same assumption as Jabbari et al. (2002, 2003), Shao et al. (2004) obtained the analytical solutions of stress fields in functionally graded circular hollow cylinder with finite length and under mechanical loading. Based on the mathematical similarity of the axisymmetric bending and buckling problems of a circular plate between the classical plate theory and Reddy's third-order shear deformation plate theory, Ma and Wang (2004) derived the analytical relations of the solutions of bending and buckling of circular FGM plate based on various plate theories, respectively, and then obtained the analytical solutions of bending and buckling of circular FGM plate. Ma and Wang (2003a,b) investigated the nonlinear bending and post-buckling behavior of functionally graded circular plate under thermal and mechanical loadings.

Cylindrical panels are widely used in engineering structures. It is a little bit easy to obtain the analytical solutions of the stress field in cylindrical panel with infinite length. However, it is difficult to derive the analytical solutions of the stress field in cylindrical panel with finite length due to the three-dimensional characteristics. Using extended power series method, Huang and Tauchert (1991) derived the analytical solutions of thermo-elastic stresses in cross-ply laminated cylindrical panels subjected to mechanical and thermal loads. Ootao and Tanigawa (2002) studied the transient thermal stresses in an angle-ply laminated composite cylindrical panel with infinite length and subjected to nonuniform heat source in the circumferential direction.

In the present work, we would like to consider a simply supported functionally graded cylindrical panel with finite length and subjected to nonuniform thermal/mechanical loads in the inner and outer surfaces of the panel. The material properties of the FGM panel are assumed to be temperature independent, vary continuously in radial direction of the panel and obey a power law of distribution of the volume fraction of the constituents. The extended power series method used by Huang and Tauchert (1991) will be employed to solve such a three-dimensional thermo-elastic problem. The paper is arranged as follows: basic equations of the problem including the thermo-elastic constitutive relations of FGM, equilibrium and heat conduction equations and corresponding temperature and mechanical boundaries are formulated in Section 2. Analytical solutions for the three-dimensional steady-state temperature and stress fields in the functionally graded cylindrical panel are derived in Sections 3 and 4, respectively. In Section 5, a mullite/molybdenum functionally graded cylindrical panel with finite length and subjected to nonaxisymmetric thermal and mechanical loads is considered as an example, and the three-dimensional temperature and stress fields are graphically presented. In Section 6, concluding remarks are presented.

2. Basic equations

A functionally graded cylindrical panel with finite length l, internal radius r_a and external radius r_b, as shown in Fig. 1, is considered. Cylindrical coordinates r, θ and z are used in analysis, where axis r is radially outward from the center of the panel, coordinate θ is in the circumference direction, and axis z is the length direction and perpendicular to the r–θ plane. The panel is simply supported at its four end edges and subjected to nonuniform steady-state thermal loads $T_a(\theta, z)$ on the inner surface, $T_b(\theta, z)$ on the outer surface and zero on the four end surfaces. The nonuniform internal pressure $q_a(\theta, z)$ and external pressure $q_b(\theta, z)$ are applied to the inner and outer surfaces of the panel.

Generally, Poisson's ratio μ of materials varies in a very small range. For simplicity, we assume μ to be a constant for functionally graded materials. Moreover, we assume the Young's modulus E, thermal expansion coefficient α and thermal conductivity coefficient λ of the FGM change continuously through the thickness of the panel and obey the following power laws (Wang and Zou, 1997):

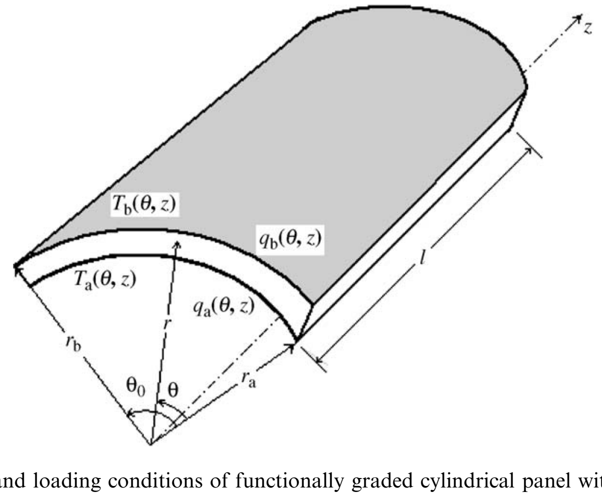

Fig. 1. Dimensions and loading conditions of functionally graded cylindrical panel with coordinate system.

$$E(r) = E_0 \left(\frac{r}{r_b}\right)^{m_1}, \tag{1a}$$

$$\alpha(r) = \alpha_0 \left(\frac{r}{r_b}\right)^{m_2}, \tag{1b}$$

$$\lambda(r) = \lambda_0 \left(\frac{r}{r_b}\right)^{m_3}, \tag{1c}$$

where E_0, α_0, λ_0, m_1, m_2 and m_3 are material constants. If $m_1 = m_2 = m_3 = 0$, the functionally graded panel reduces to a homogeneous panel.

In the cylindrical coordinate system, thermo-elastic constitutive relations of functionally graded materials are as follows:

$$\sigma_r = \frac{E(r)}{(1+\mu)(1-2\mu)}[(1-\mu)\varepsilon_r + \mu\varepsilon_\theta + \mu\varepsilon_z] - \frac{\alpha(r)E(r)}{1-2\mu}T(r,\theta,z), \tag{2a}$$

$$\sigma_\theta = \frac{E(r)}{(1+\mu)(1-2\mu)}[\mu\varepsilon_r + (1-\mu)\varepsilon_\theta + \mu\varepsilon_z] - \frac{\alpha(r)E(r)}{1-2\mu}T(r,\theta,z), \tag{2b}$$

$$\sigma_z = \frac{E(r)}{(1+\mu)(1-2\mu)}[\mu\varepsilon_r + \mu\varepsilon_\theta + (1-\mu)\varepsilon_z] - \frac{\alpha(r)E(r)}{1-2\mu}T(r,\theta,z), \tag{2c}$$

$$\tau_{\theta z} = \frac{E(r)}{2(1+\mu)}\gamma_{\theta z}, \tag{2d}$$

$$\tau_{zr} = \frac{E(r)}{2(1+\mu)}\gamma_{zr}, \tag{2e}$$

$$\tau_{r\theta} = \frac{E(r)}{2(1+\mu)}\gamma_{r\theta}, \tag{2f}$$

where $T(r,\theta,z)$ is temperature filed and the components of strain can be calculated from the following geometric relations:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + u\right), \quad \varepsilon_z = \frac{\partial w}{\partial z}, \tag{3a}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}, \quad \gamma_{zr} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \tag{3b}$$

with u, v and w being the displacements in r, θ and z directions, respectively. The stress components should satisfy the following equilibrium equations:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{4a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0, \tag{4b}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0. \tag{4c}$$

If nonuniform steady-state thermal loads $T_a(\theta,z)$ and $T_b(\theta,z)$ are applied to the inner and outer surfaces of the panel, respectively, then three-dimensional heat conduction obeys the following equation:

$$\left[\frac{\partial^2}{\partial r^2} + \left(\frac{1}{\lambda(r)}\frac{\mathrm{d}\lambda(r)}{\mathrm{d}r} + \frac{1}{r}\right)\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right]T = 0. \tag{5}$$

For the present problem, the temperature boundary conditions can be expressed as

$$T(r,0,z) = T(r,\theta_0,z) = 0, \tag{6a}$$

$$T(r,\theta,0) = T(r,\theta,l) = 0, \tag{6b}$$

$$T(r_a,\theta,z) = T_a(\theta,z), \tag{6c}$$

$$T(r_b,\theta,z) = T_b(\theta,z) \tag{6d}$$

and the boundary conditions of displacements and stresses can be expressed as

$$\left.\begin{aligned} &u(r,0,z) = w(r,0,z) = 0,\\ &\sigma_\theta(r,0,z) = \tau_{r\theta}(r,0,z) = \tau_{\theta z}(r,0,z) = 0, \end{aligned}\right\} \tag{7a}$$

$$\left.\begin{aligned} &u(r,\theta_0,z) = w(r,\theta_0,z) = 0,\\ &\sigma_\theta(r,\theta_0,z) = \tau_{r\theta}(r,\theta_0,z) = \tau_{\theta z}(r,\theta_0,z) = 0, \end{aligned}\right\} \tag{7b}$$

$$\left.\begin{aligned} &u(r,\theta,0) = v(r,\theta,0) = 0,\\ &\sigma_z(r,\theta,0) = \tau_{rz}(r,\theta,0) = \tau_{\theta z}(r,\theta,0) = 0, \end{aligned}\right\} \tag{7c}$$

$$\left.\begin{aligned} &u(r,\theta,l) = v(r,\theta,l) = 0,\\ &\sigma_z(r,\theta,l) = \tau_{rz}(r,\theta,l) = \tau_{\theta z}(r,\theta,l) = 0, \end{aligned}\right\} \tag{7d}$$

$$\sigma_r(r_a,\theta,z) = q_a(\theta,z), \tau_{rz}(r_a,\theta,z) = \tau_{r\theta}(r_a,\theta,z) = 0, \tag{7e}$$

$$\sigma_r(r_b,\theta,z) = q_b(\theta,z), \tau_{rz}(r_b,\theta,z) = \tau_{r\theta}(r_b,\theta,z) = 0. \tag{7f}$$

Now a three-dimensional boundary problem has been formulated. To simplify the solving process, we introduce the following dimensionless variables:

$$\begin{aligned}
&R = \frac{r}{r_b}, \quad Z = \frac{z}{r_b}, \quad R_a = \frac{r_a}{r_b}, \quad R_b = \frac{r_b}{r_b} = 1, \quad L = \frac{l}{r_b},\\
&Y = \frac{E}{E_0}, \quad \Omega = \frac{\alpha}{\alpha_0}, \quad \kappa = \frac{\lambda}{\lambda_0},\\
&\Theta = \frac{T}{T_0}, \quad \Theta_a = \frac{T_a}{T_0}, \quad \Theta_b = \frac{T_b}{T_0},\\
&U = \frac{u}{\alpha_0 T_0 r_b}, \quad V = \frac{v}{\alpha_0 T_0 r_b}, \quad W = \frac{w}{\alpha_0 T_0 r_b},\\
&e_{rr} = \frac{\varepsilon_r}{\alpha_0 T_0}, \quad e_{\theta\theta} = \frac{\varepsilon_\theta}{\alpha_0 T_0}, \quad e_{zz} = \frac{\varepsilon_z}{\alpha_0 T_0},\\
&e_{r\theta} = \frac{\gamma_{r\theta}}{\alpha_0 T_0}, \quad e_{rz} = \frac{\gamma_{rz}}{\alpha_0 T_0}, \quad e_{z\theta} = \frac{\gamma_{z\theta}}{\alpha_0 T_0},\\
&\Sigma_r = \frac{\sigma_r}{\alpha_0 E_0 T_0}, \quad \Sigma_\theta = \frac{\sigma_\theta}{\alpha_0 E_0 T_0}, \quad \Sigma_z = \frac{\sigma_z}{\alpha_0 E_0 T_0},\\
&\Sigma_{r\theta} = \frac{\tau_{r\theta}}{\alpha_0 E_0 T_0}, \quad \Sigma_{rz} = \frac{\tau_{rz}}{\alpha_0 E_0 T_0}, \quad \Sigma_{\theta z} = \frac{\tau_{\theta z}}{\alpha_0 E_0 T_0},\\
&Q_a = \frac{q_a}{\alpha_0 E_0 T_0}, \quad Q_b = \frac{q_b}{\alpha_0 E_0 T_0},
\end{aligned}$$

where T_0 is a reference value of temperature. Then, the dimensionless equilibrium and heat conduction equations can be expressed as

$$\left[\frac{\partial^2}{\partial R^2}+(m_1+1)\frac{1}{R}\frac{\partial}{\partial R}+\left(\frac{\mu m_1}{1-\mu}-1\right)\frac{1}{R^2}+\frac{1-2\mu}{2-2\mu}\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}+\frac{1-2\mu}{2-2\mu}\frac{\partial^2}{\partial Z^2}\right]U$$
$$+\left[\frac{1}{2-2\mu}\frac{1}{R}\frac{\partial^2}{\partial R\partial\theta}+\frac{(4+2m_1)\mu-3}{2-2\mu}\frac{1}{R^2}\frac{\partial}{\partial\theta}\right]V+\left[\frac{1}{2-2\mu}\frac{\partial^2}{\partial R\partial Z}+\frac{\mu m_1}{1-\mu}\frac{1}{R}\frac{\partial}{\partial Z}\right]W$$
$$-\frac{1+\mu}{1-\mu}\left[\Omega\frac{\partial}{\partial R}+\frac{1}{Y}\frac{\mathrm{d}(\Omega Y)}{\mathrm{d}R}\right]\Theta=0, \tag{8a}$$

$$\left[\frac{1}{1-2\mu}\frac{1}{R}\frac{\partial^2}{\partial R\partial\theta}+\left(m_1+\frac{3-4\mu}{1-2\mu}\right)\frac{1}{R^2}\frac{\partial}{\partial\theta}\right]U$$
$$+\left[\frac{\partial^2}{\partial R^2}+(m_1+1)\frac{1}{R}\frac{\partial}{\partial R}-(m_1+1)\frac{1}{R^2}+\frac{2-2\mu}{1-2\mu}\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial Z^2}\right]V$$
$$+\frac{1}{1-2\mu}\frac{1}{R}\frac{\partial^2 W}{\partial\theta\partial Z}-\frac{2+2\mu}{1-2\mu}\Omega\frac{1}{R}\frac{\partial\Theta}{\partial\theta}=0, \tag{8b}$$

$$\left[\frac{1}{1-2\mu}\frac{\partial^2}{\partial R\partial Z}+\left(m_1+\frac{1}{1-2\mu}\right)\frac{1}{R}\frac{\partial}{\partial Z}\right]U+\frac{1}{1-2\mu}\frac{1}{R}\frac{\partial^2 V}{\partial\theta\partial Z}$$
$$+\left[\frac{\partial^2}{\partial R^2}+(m_1+1)\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}+\frac{2-2\mu}{1-2\mu}\frac{\partial^2}{\partial Z^2}\right]W-\frac{2+2\mu}{1-\mu}\Omega\frac{\partial\Theta}{\partial Z}=0 \tag{8c}$$

and

$$\left[\frac{\partial^2}{\partial R^2}+\left(\frac{1}{\kappa(R)}\frac{\mathrm{d}\kappa(R)}{\mathrm{d}R}+\frac{1}{R}\right)\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial Z^2}\right]\Theta=0. \tag{9}$$

The temperature boundary conditions (6) can be rewritten as

$$\Theta(R,0,Z)=\Theta(R,\theta_0,Z)=0, \tag{10a}$$
$$\Theta(R,\theta,0)=\Theta(R,\theta,L)=0, \tag{10b}$$
$$\Theta(R_a,\theta,Z)=\Theta_a(\theta,Z), \tag{10c}$$
$$\Theta(R_b,\theta,Z)=\Theta_b(\theta,Z). \tag{10d}$$

The displacement and stress boundary conditions (7) can be rewritten as

$$\left.\begin{array}{l}U(R,0,Z)=W(R,0,Z)=0,\\ \Sigma_\theta(R,0,Z)=\Sigma_{r\theta}(R,0,Z)=\Sigma_{\theta z}(R,0,Z)=0,\end{array}\right\} \tag{11a}$$
$$\left.\begin{array}{l}U(R,\theta_0,Z)=W(R,\theta_0,Z)=0,\\ \Sigma_\theta(R,\theta_0,Z)=\Sigma_{r\theta}(R,\theta_0,Z)=\Sigma_{\theta z}(R,\theta_0,Z)=0,\end{array}\right\} \tag{11b}$$
$$\left.\begin{array}{l}U(R,\theta,0)=V(R,\theta,0)=0,\\ \Sigma_z(R,\theta,0)=\Sigma_{rz}(R,\theta,0)=\Sigma_{\theta z}(R,\theta,0)=0,\end{array}\right\} \tag{11c}$$
$$\left.\begin{array}{l}U(R,\theta,L)=V(R,\theta,L)=0,\\ \Sigma_z(R,\theta,L)=\Sigma_{rz}(R,\theta,L)=\Sigma_{\theta z}(R,\theta,L)=0,\end{array}\right\} \tag{11d}$$
$$\Sigma_r(R_a,\theta,Z)=Q_a(\theta,Z),\quad \Sigma_{rz}(R_a,\theta,Z)=\Sigma_{r\theta}(R_a,\theta,Z)=0, \tag{11e}$$
$$\Sigma_r(R_b,\theta,z)=Q_b(\theta,Z),\quad \Sigma_{rz}(R_b,\theta,Z)=\Sigma_{r\theta}(R_b,\theta,Z)=0. \tag{11f}$$

In what follows, the extended power series method used by Huang and Tauchert (1991) will be employed to derive the analytical solutions of the steady-state temperature and stress fields for the three-dimensional problem formulated above.

3. Analytical solution for the three-dimensional steady-state temperature field

From Eq. (1c) one can obtain the dimensionless thermal conductivity coefficient κ, then substituting into Eq. (9) yields

$$\left[\frac{\partial^2}{\partial R^2}+(m_3+1)\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial Z^2}\right]\Theta=0. \tag{12}$$

Using Navier trigonometric series, the solution of Eq. (12) satisfying the temperature boundary conditions (10a) and (10b) can be assumed as

$$\Theta(R,\theta,Z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Theta_{nm}(R)\sin(b\theta)\sin(aZ), \tag{13}$$

where $\Theta_{nm}(R)$ are unknown functions, $a=n\pi/L$ and $b=m\pi/\theta_0$.

Substituting Eq. (13) into Eqs. (12), (10c) and (10d), respectively, we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}R^2}+(m_3+1)\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}-\left(\frac{b^2}{R^2}+a^2\right)\right]\Theta_{nm}(R)=0 \tag{14}$$

and

$$\Theta(R_a)=\Theta_{anm}, \tag{15a}$$

$$\Theta(R_b)=\Theta_{bnm}, \tag{15b}$$

where

$$\Theta_{anm}=\frac{4}{\theta_0 L}\int_0^L\int_0^{\theta_0}\Theta_a(\theta,Z)\sin(b\theta)\sin(aZ)\,\mathrm{d}\theta\,\mathrm{d}Z,$$

$$\Theta_{bnm}=\frac{4}{\theta_0 L}\int_0^L\int_0^{\theta_0}\Theta_b(\theta,Z)\sin(b\theta)\sin(aZ)\,\mathrm{d}\theta\,\mathrm{d}Z.$$

It is clear that the point $R=0$ is a regular singular point of Eq. (14). Such that Frobenius method (Myint-U, 1978) can be employed to solve the ordinary differential Eq. (14).

It is assumed that the solution of Eq. (14) has the following form expressed in terms of the extended power series:

$$\Theta_{nm}(R)=\sum_{k=0}^{\infty}A_k R^{\eta+k}, \tag{16}$$

where A_k are coefficients to be determined and η is an exponent.

Substituting Eq. (16) into Eq. (14) and letting the coefficient of $R^{\eta+k}$ equal to zero, one then obtains the following recurrence relation:

$$A_k\left[(\eta+k)^2+m_3(\eta+k)-b^2\right]=A_{k-2}a^2,\quad k=0,1,2,\ldots, \tag{17}$$

where $A_{-1}=0$ and $A_{-2}=0$. Since $A_0\neq 0$, we can obtain the indicial equation of Eq. (14) for $k=0$,

$$\eta^2+m_3\eta-b^2=0. \tag{18}$$

From Eq. (18) one can obtain the solutions of η. To simplify analysis, we consider the case of distinctive real roots of Eq. (18). Two roots of η are denoted to η_1 and η_2. Such that we obtain the following solution of $\Theta_{nm}(R)$:

$$\Theta_{nm}(R) = \sum_{k=0}^{\infty} \left(\beta_1 A_{k1} R^{\eta_1 + k} + \beta_2 A_{k2} R^{\eta_2 + k} \right), \tag{19}$$

where β_1 and β_2 are unknown constants and can be determined from Eqs. (15a) and (15b), A_{k1} and A_{k2} are constants corresponding to η_1 and η_2, respectively. Without losing generality, we assume $A_{01} = 1$ and $A_{02} = 1$ for $k = 0$. For $k \geqslant 1$, A_{k1} and A_{k2} can be obtained from the recurrence relation (17).

Substituting Eq. (19) into Eq. (13), one then obtains the analytical solution for the three-dimensional steady-state temperature field in the functionally graded cylindrical panel with finite length. In what follows, we will derive the analytical solutions for the stress field in the functionally graded cylindrical panel.

4. Analytical solutions for the stress field

Using Navier trigonometric series, solutions of Eqs. (8) satisfying the boundary conditions (11a)–(11d) can be assumed as

$$U(R, \theta, Z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} U_{nm}(R) \sin(b\theta) \sin(aZ), \tag{20a}$$

$$V(R, \theta, Z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} V_{nm}(R) \cos(b\theta) \sin(aZ), \tag{20b}$$

$$W(R, \theta, Z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} W_{nm}(R) \sin(b\theta) \cos(aZ). \tag{20c}$$

Substituting Eqs. (13) and (20) into Eqs. (8) and boundary conditions (11e) and (11f), we have

$$\begin{aligned} &\left[\frac{\mathrm{d}^2}{\mathrm{d}R^2} + (m_1 + 1) \frac{1}{R} \frac{\mathrm{d}}{\mathrm{d}R} + \left(\frac{\mu m_1}{1 - \mu} - \frac{1 - 2\mu}{2 - 2\mu} b^2 - 1 \right) \frac{1}{R^2} - \frac{1 - 2\mu}{2 - 2\mu} a^2 \right] U_{nm}(R) \\ &\quad - \left[\frac{b}{2 - 2\mu} \frac{1}{R} \frac{\mathrm{d}}{\mathrm{d}R} + \frac{(4 + 2m_1)\mu - 3}{2 - 2\mu} \frac{b}{R^2} \right] V_{nm}(R) - \left[\frac{a}{2 - 2\mu} \frac{\mathrm{d}}{\mathrm{d}R} + \frac{\mu m_1}{1 - \mu} \frac{a}{R} \right] W_{nm}(R) \\ &\qquad = \frac{1 + \mu}{1 - \mu} \left[\Omega \frac{\mathrm{d}}{\mathrm{d}R} + \frac{1}{Y} \frac{\mathrm{d}(\Omega Y)}{\mathrm{d}R} \right] \Theta_{nm}(R), \end{aligned} \tag{21a}$$

$$\begin{aligned} &\left[\frac{b}{1 - 2\mu} \frac{1}{R} \frac{\mathrm{d}}{\mathrm{d}R} + \left(m_1 + \frac{3 - 4\mu}{1 - 2\mu} \right) \frac{b}{R^2} \right] U_{nm} - \frac{ab}{1 - 2\mu} \frac{1}{R} W_{nm} \\ &\quad + \left[\frac{\mathrm{d}^2}{\mathrm{d}R^2} + (m_1 + 1) \frac{1}{R} \frac{\mathrm{d}}{\mathrm{d}R} + \left(\frac{2 - 2\mu}{1 - 2\mu} b^2 + m_1 + 1 \right) \frac{1}{R^2} - a^2 \right] V_{nm} = \frac{2 + 2\mu}{1 - 2\mu} \Omega \Theta_{nm}, \end{aligned} \tag{21b}$$

$$\begin{aligned} &\left[\frac{a}{1 - 2\mu} \frac{\mathrm{d}}{\mathrm{d}R} + \left(m_1 + \frac{1}{1 - 2\mu} \right) \frac{a}{R} \right] U_{nm}(R) - \frac{ab}{1 - 2\mu} \frac{1}{R} V_{nm}(R) \\ &\quad + \left[\frac{\mathrm{d}^2}{\mathrm{d}R^2} + (m_1 + 1) \frac{1}{R} \frac{\mathrm{d}}{\mathrm{d}R} - \frac{b^2}{R^2} - \frac{2 - 2\mu}{1 - 2\mu} a^2 \right] W_{nm}(R) = \frac{2 + 2\mu}{1 - \mu} \Omega \Theta_{nm}(R) \end{aligned} \tag{21c}$$

and

$$(1-\mu)\frac{\mathrm{d}U_{nm}(R_a)}{\mathrm{d}R}-\frac{\mu b}{R}V_{nm}(R_a)+\frac{\mu}{R}U_{nm}(R_a)-\mu aW_{nm}(R_a)-(1+\mu)\Omega(R_a)\Theta_{nm}(R_a)=Q_{anm}, \tag{22a}$$

$$\frac{b}{R}U_{nm}(R_a)+\frac{\mathrm{d}V_{nm}(R_a)}{\mathrm{d}R}-\frac{V_{nm}(R_a)}{R}=0, \tag{22b}$$

$$\frac{\mathrm{d}W_{nm}(R_a)}{\mathrm{d}R}+aU_{nm}(R_a)=0, \tag{22c}$$

$$(1-\mu)\frac{\mathrm{d}U_{nm}(R_b)}{\mathrm{d}R}-\frac{\mu b}{R}V_{nm}(R_b)+\frac{\mu}{R}U_{nm}(R_b)-\mu aW_{nm}-(1+\mu)\Omega(R_b)\cdot\Theta_{nm}(R_b)=Q_{bnm}, \tag{22d}$$

$$\frac{b}{R}U_{nm}(R_b)+\frac{\mathrm{d}V_{nm}(R_b)}{\mathrm{d}R}-\frac{V_{nm}(R_b)}{R}=0, \tag{22e}$$

$$\frac{\mathrm{d}W_{nm}(R_b)}{\mathrm{d}R}+aU_{nm}(R_b)=0, \tag{22f}$$

where

$$Q_{anm}=\frac{(1+\mu)(1-2\mu)}{Y}\frac{4}{\theta_0 L}\int_0^L\int_0^{\theta_0}Q_a(\theta,Z)\sin(b\theta)\sin(aZ)\,\mathrm{d}\theta\,\mathrm{d}Z,$$
$$Q_{bnm}=\frac{(1+\mu)(1-2\mu)}{Y}\frac{4}{\theta_0 L}\int_0^L\int_0^{\theta_0}Q_b(\theta,Z)\sin(b\theta)\sin(aZ)\,\mathrm{d}\theta\,\mathrm{d}Z.$$

It is clear that the point $R = 0$ is a regular singular point of Eqs. (21). Similar to Section 3, Frobenius method (Myint-U, 1978) can be employed to derive the solutions of Eqs. (21). It is assumed that the homogeneous solutions of Eqs. (21) have the following forms expressed in terms of the extended power series:

$$U^g_{nm}(R)=\sum_{k=0}^{\infty}B_kR^{s+k}, \tag{23a}$$

$$V^g_{nm}(R)=\sum_{k=0}^{\infty}C_kR^{s+k}, \tag{23b}$$

$$W^g_{nm}(R)=\sum_{k=0}^{\infty}D_kR^{s+k}, \tag{23c}$$

where s is exponent, B_k, C_k, and D_k are coefficients to be determined. Substituting Eqs. (23) into Eqs. (21) and letting the coefficients of R^{s+k} equal to zero, one then obtains the following recurrence equations:

$$L_1B_k+M_1C_k=P_1B_{k-2}+R_1D_{k-1},\quad k=0,1,2,\ldots, \tag{24a}$$
$$L_2B_k+M_2C_k=Q_2C_{k-2}+R_2D_{k-1},\quad k=0,1,2,\ldots, \tag{24b}$$
$$N_3D_k=P_3B_{k-1}+Q_3C_{k-1}+R_3D_{k-2},\quad k=0,1,2,\ldots, \tag{24c}$$

where

$$L_1=(s+k)^2+m_1(s+k)-\frac{2-2\mu}{1-2\mu}b^2+\frac{m_1\mu}{1-\mu}-1,$$

$$L_2=\frac{b}{1-2\mu}(s+k)+\left(m_1+\frac{3-4\mu}{1-2\mu}\right)b,$$

$$M_1=-\frac{b}{2-2\mu}(s+k)-\frac{(2m_1+4)\mu-3}{2-2\mu},$$

$$M_2 = (s+k)^2 + m_1(s+k) - (m_1+1) - \frac{2-2\mu}{1-2\mu}b^2,$$

$$N_3 = (s+k)^2 + m_1(s+k) - b^2,$$

$$P_1 = \frac{1-2\mu}{2-2\mu}a^2,$$

$$P_3 = -\frac{a}{1-2\mu}(s+k) - m_1 a,$$

$$Q_2 = a^2,$$

$$Q_3 = \frac{ab}{1-2\mu},$$

$$R_1 = \frac{a}{2-2\mu}(s+k-1) + \frac{m_1\mu a}{1-\mu},$$

$$R_2 = \frac{ab}{1-2\mu},$$

$$R_3 = \frac{2-2\mu}{1-2\mu}a^2.$$

Since $B_0 \neq 0$, $C_0 \neq 0$ and $D_0 \neq 0$, we can obtain the indicial equations of Eqs. (21) for $k = 0$

$$s^4 + As^3 + Bs^2 + Cs + D = 0, \tag{25a}$$

$$s^2 + m_1 s - b^2 = 0, \tag{25b}$$

where

$$A = 2m_1,$$

$$B = m_1^2 - \frac{1-2\mu}{1-\mu}m_1 - 2b^2 - 2,$$

$$C = \frac{2\mu-1}{1-\mu}m_1^2 - (2b^2+2)m_1,$$

$$D = \left(\frac{1-2\mu}{2-2\mu}b^2 - \frac{m_1\mu}{1-\mu} - 1\right)\left(\frac{2-2\mu}{1-2\mu}b^2 + m_1 + 1\right) + \frac{(2m_1+4)\mu-3}{2-2\mu}\left(m_1 + \frac{3-4\mu}{1-2\mu}\right)b^2.$$

From Eqs. (25) one can obtain the solutions of s. To simplify analysis, we consider the case of distinctive real roots of Eqs. (25). The roots of Eq. (25a) are denoted as s_1, s_2, s_3 and s_4. The roots of Eq. (25b) are denoted as s_5 and s_6. Such that the homogeneous solutions of Eqs. (21) can be expressed as

$$\left\{U^g_{nm}, \quad V^g_{nm}, \quad W^g_{nm}\right\} = \sum_{j=1}^{6}\sum_{k=0}^{\infty}\zeta_j\left\{B_{jk}, \quad C_{jk}, \quad D_{jk}\right\}R^{s_j+k}, \tag{26}$$

where ζ_j are unknown constants which can be determined from Eqs. (22). $\{B_{jk}, C_{jk}, D_{jk}\}$ are eigenvectors corresponding to eigenvalue s_j.

Without losing the generality, in the case of $k = 0$, we assume,

$$B_{j0} = 1, \quad C_{j0} = H, \quad D_{j0} = 0 \quad (j = 1,2,3,4),$$

$$B_{j0} = 0, \quad C_{j0} = 0, \quad D_{j0} = 1 \quad (j = 5,6),$$

where H is a derived quantity which can be determined from Eqs. (24a) and (24b), i.e.

$$H = \frac{(2-2\mu)(s_j^2 + m_1 s_j) + 2(m_1+1)\mu - (1-2\mu)b^2 - 2}{b s_j + (2m_1+4)\mu b - 3b}.$$

For $k \geqslant 1$, the eigenvectors $\{B_{jk}, C_{jk}, D_{jk}\}$ can be derived from the recurrence relations (24).

In what follows, the method used by Ootao and Tanigawa (2002) is employed to derive the special solutions of Eqs. (21). Considering Eq. (19), the special solutions of Eqs. (21) can be assumed as

$$U^p_{nm}(R) = \sum_{k=0}^{\infty} \left(B_{k1} R^{\eta_1+k+m_2+1} + B_{k2} R^{\eta_2+k+m_2+1}\right), \tag{27a}$$

$$V^p_{nm}(R) = \sum_{k=0}^{\infty} \left(C_{k1} R^{\eta_1+k+m_2+1} + C_{k2} R^{\eta_2+k+m_2+1}\right), \tag{27b}$$

$$W^p_{nm}(R) = \sum_{k=0}^{\infty} \left(D_{k1} R^{\eta_1+k+m_2+1} + D_{k2} R^{\eta_2+k+m_2+1}\right). \tag{27c}$$

Substituting Eqs. (27) into Eqs. (21) and comparing the coefficients of $R^{\eta_1+k+m_2-1}$ and $R^{\eta_2+k+m_2-1}$, respectively, we obtain

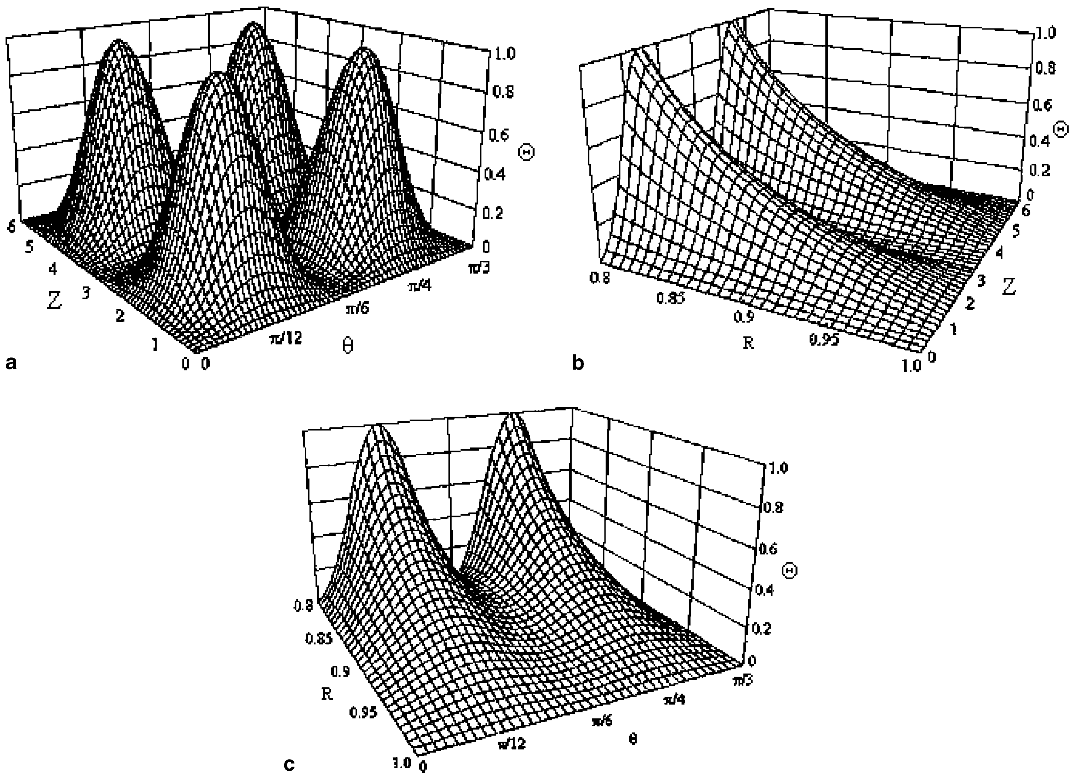

Fig. 2. Distribution of dimensionless temperature: (a) internal surface of the panel, i.e. $R = 0.8$, (b) section $\theta = \pi/4$ and (c) section $Z = 1.5$.

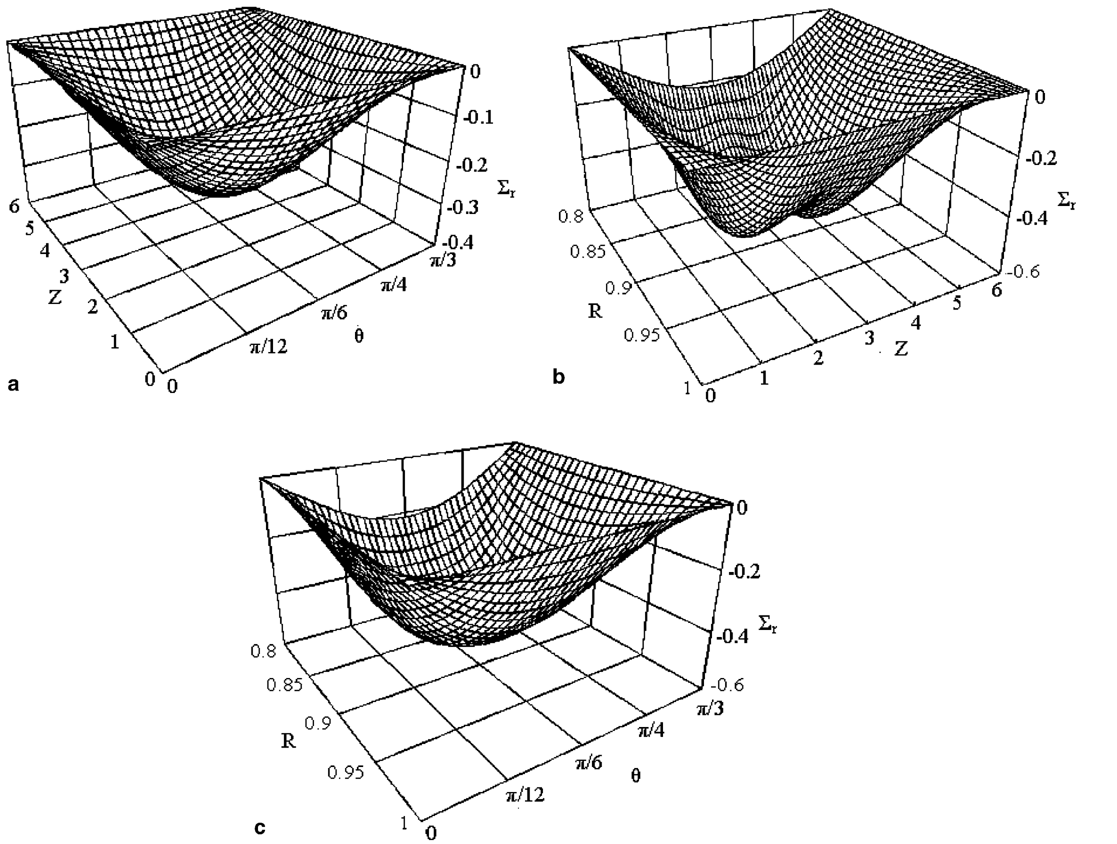

Fig. 3. Distributions of dimensionless radial stress: (a) at section: $R = 0.8$, (b) at section $\theta = \pi/4$ and (c) at section $Z = 3/2$.

$$\begin{aligned}
&\left[(\eta_1 + k + m_2 + 1)(\eta_1 + k + m_2 + m_1 + 1) - \frac{1-2\mu}{2-2\mu}b^2 + \frac{\mu m_1}{1-\mu} - 1\right]B_{k1} \\
&\quad - \frac{1-2\mu}{2-2\mu}a^2 B_{k-2,1} - b\left[\frac{1}{2-2\mu}(\eta_1 + k + m_2 + 1) + \frac{(2m_1+4)\mu - 3}{2-2\mu}\right]C_{k1} \\
&\quad - a\left[\frac{\eta_1 + k + m_2}{2-2\mu} + \frac{m_1\mu}{1-\mu}\right]D_{k-1,1} - \frac{1+\mu}{1-\mu}\alpha_0\beta_1[(\eta_1 + k + m_2)A_{k1} + m_1 A_{k-1,1}] = 0,
\end{aligned} \tag{28a}$$

$$\begin{aligned}
&b\left[\frac{1}{1-2\mu}(\eta_1 + k + m_2 + 1) + m_1 + \frac{3-4\mu}{1-2\mu}\right]B_{k1} - a^2 C_{k-2,1} \\
&\quad + \left[(\eta_1 + k + m_2 + 1)(\eta_1 + k + m_2 + m_1 + 1) - \frac{1-2\mu}{2-2\mu}b^2 - m_1 - 1\right]C_{k1} \\
&\quad - \frac{ab}{1-2\mu}D_{k-1,1} - \frac{2+2\mu}{1-2\mu}b\alpha_0\beta_1 A_{k1} = 0,
\end{aligned} \tag{28b}$$

$$\begin{aligned}
&a\left[\frac{1}{1-2\mu}(\eta_1 + k + m_2) + m_1 + \frac{1}{1-2\mu}\right]B_{k-1,1} \\
&\quad - \frac{ab}{1-2\mu}C_{k-1,1} + \left[(\eta_1 + k + m_2 + 1)(\eta_1 + k + m_2 + m_1 + 1) - b^2\right]D_{k1} \\
&\quad - \frac{2-2\mu}{1-2\mu}a^2 D_{k-2,1} - \frac{2+2\mu}{1-2\mu}a\alpha_0\beta_1 A_{k-1,1} = 0
\end{aligned} \tag{28c}$$

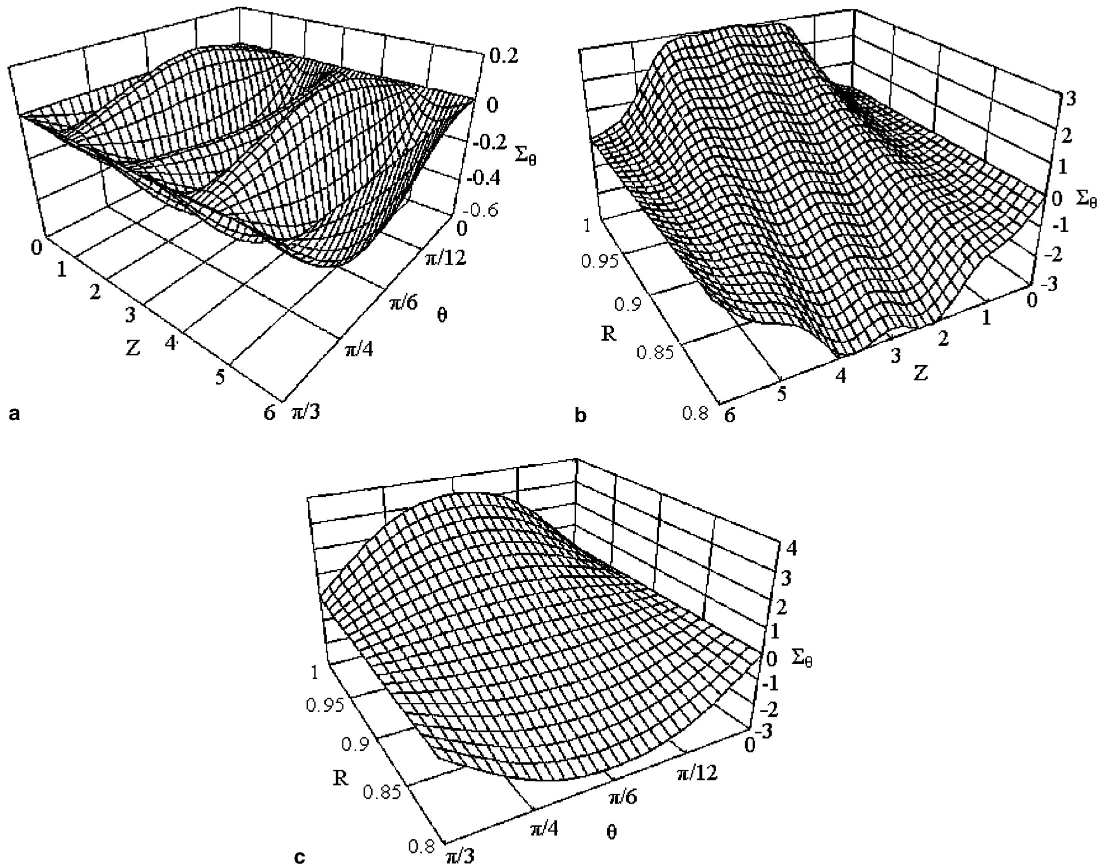

Fig. 4. Distribution of dimensionless circumferential stress Σ_θ: (a) section $R = 0.9$, (b) section $\theta = \pi/4$ and (c) section $Z = 1.5$.

and

$$\left[(\eta_2 + k + m_2 + 1)(\eta_2 + k + m_2 + m_1 + 1) - \frac{1-2\mu}{2-2\mu}b^2 + \frac{\mu m_1}{1-\mu} - 1\right]B_{k2} - \frac{1-2\mu}{2-2\mu}a^2 B_{k-2,2} - b\left[\frac{1}{2-2\mu}(\eta_2 + k + m_2 + 1) + \frac{(2m_1+4)\mu - 3}{2-2\mu}\right]C_{k2} - a\left[\frac{\eta_2 + k + m_2}{2-2\mu} + \frac{m_1\mu}{1-\mu}\right]D_{k-1,2} - \frac{1+\mu}{1-\mu}\alpha_0\beta_2[(\eta_1 + k + m_2)A_{k2} + m_1 A_{k-1,2}] = 0, \tag{29a}$$

$$b\left[\frac{1}{1-2\mu}(\eta_2 + k + m_2 + 1) + m_1 + \frac{3-4\mu}{1-2\mu}\right]B_{k2} - a^2 C_{k-2,2} + \left[(\eta_2 + k + m_2 + 1)(\eta_2 + k + m_2 + m_1 + 1) - \frac{1-2\mu}{2-2\mu}b^2 - m_1 - 1\right]C_{k2} - \frac{ab}{1-2\mu}D_{k-1,2} - \frac{2+2\mu}{1-2\mu}b\alpha_0\beta_2 A_{k2} = 0, \tag{29b}$$

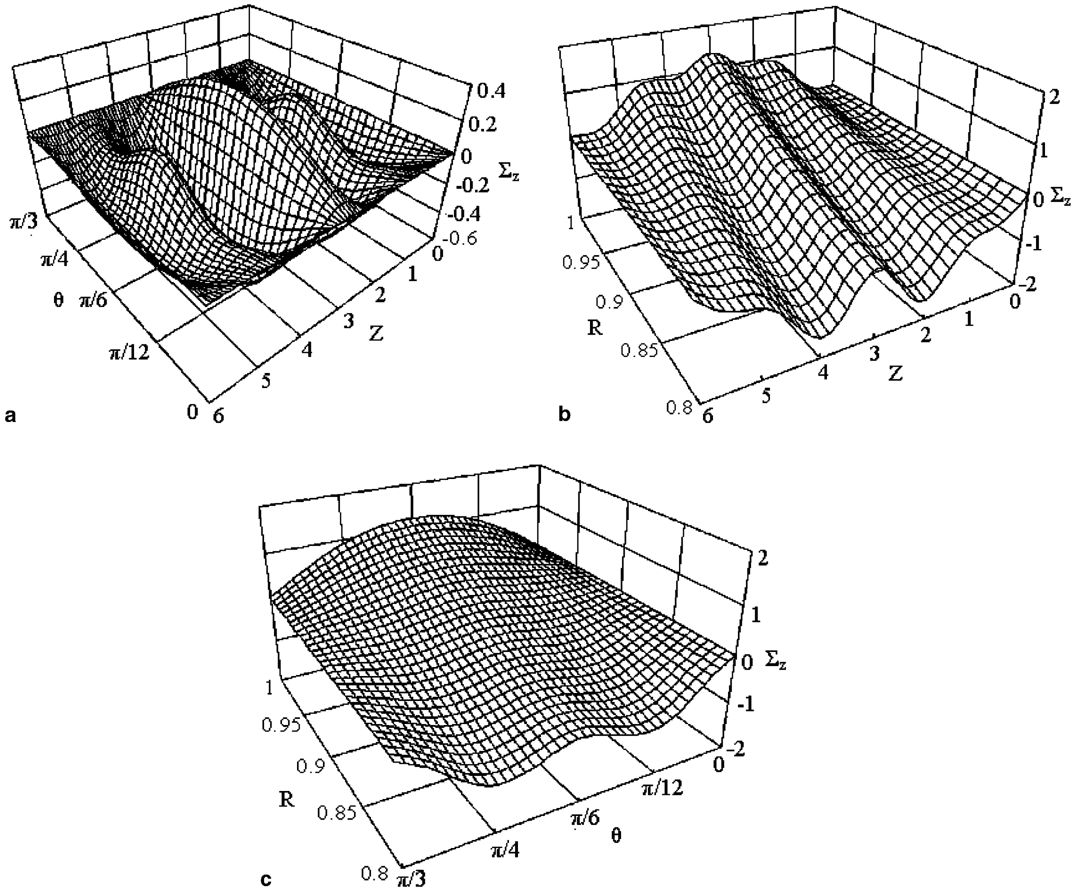

Fig. 5. Distribution of dimensionless axial stress Σ_z: (a) section $R = 0.9$, (b) section $\theta = \pi/4$ and (c) section $Z = 1.5$.

$$a\left[\frac{1}{1-2\mu}(\eta_2+k+m_2)+m_1+\frac{1}{1-2\mu}\right]B_{k-1,2}-\frac{ab}{1-2\mu}C_{k-1,2}$$
$$+\left[(\eta_2+k+m_2+1)(\eta_2+k+m_2+m_1+1)-b^2\right]D_{k2}-\frac{2-2\mu}{1-2\mu}a^2D_{k-2,2}$$
$$-\frac{2+2\mu}{1-2\mu}a\alpha_0\beta_2A_{k-1,2}=0, \quad (29c)$$

where

$$A_{-2,1}=A_{-1,1}=0, \quad A_{-2,2}=A_{-1,2}=0,$$
$$B_{-2,1}=B_{-1,1}=0, \quad B_{-2,2}=B_{-1,2}=0,$$
$$C_{-2,1}=C_{-1,1}=0, \quad C_{-2,2}=C_{-1,2}=0,$$
$$D_{-2,1}=D_{-1,1}=0, \quad D_{-2,2}=D_{-1,2}=0.$$

Letting $k=0$, we can obtain the coefficients B_{01}, B_{02}, C_{01}, C_{02}, D_{01} and D_{02} from Eqs. (28) and (29), respectively. For $k \geqslant 1$, one can derive the coefficients B_{k1}, B_{k2}, C_{k1}, C_{k2}, D_{k1} and D_{k2} expressed with B_{01}, B_{02}, C_{01}, C_{02}, D_{01} and D_{02} from the above recurrence Eqs. (28) and (29). From Eqs. (26) and (27), we obtain the solutions of Eqs. (21)

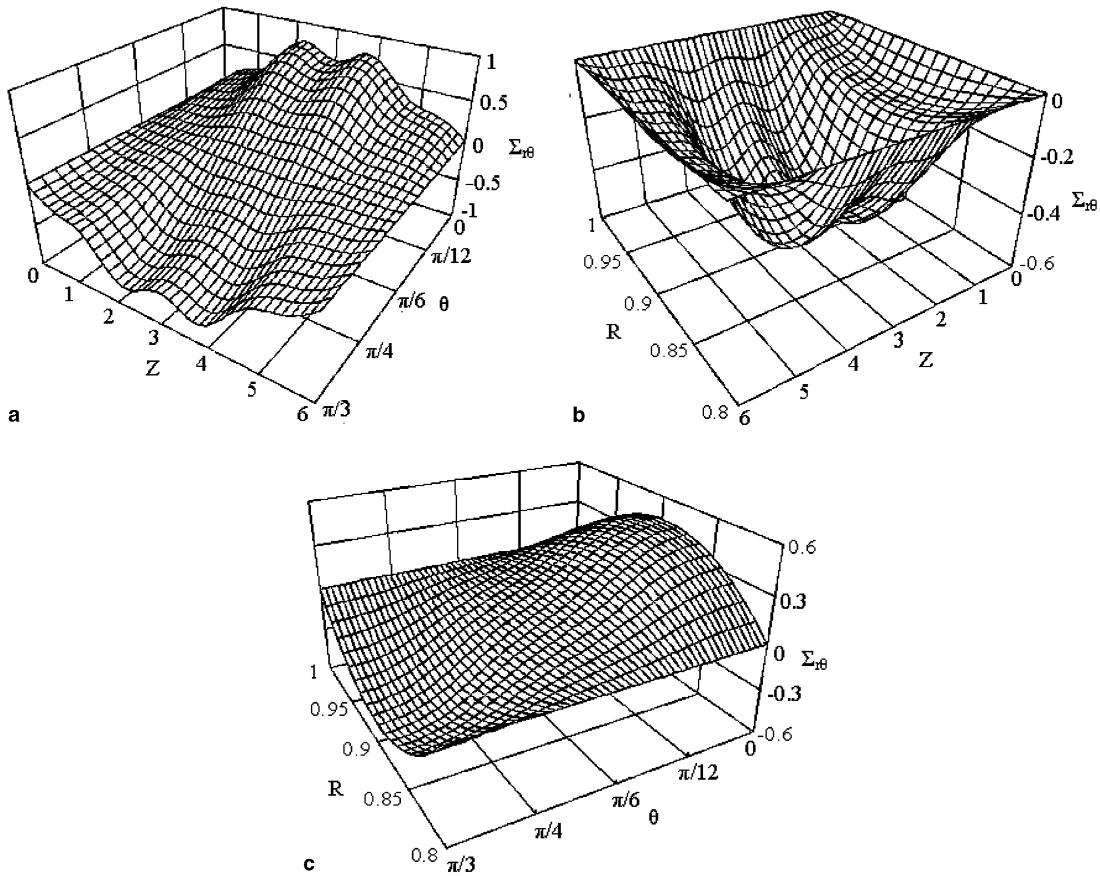

Fig. 6. Distribution of dimensionless shear stress $\Sigma_{r\theta}$: (a) section $R = 0.9$, (b) section $\theta = \pi/4$ and (c) section $Z = 1.5$.

$$U_{nm}(R) = U^{g}_{nm}(R) + U^{p}_{nm}(R), \tag{30a}$$
$$V_{nm}(R) = V^{g}_{nm}(R) + V^{p}_{nm}(R), \tag{30b}$$
$$W_{nm}(R) = W^{g}_{nm}(R) + W^{p}_{nm}(R), \tag{30c}$$

Substituting Eqs. (30) into Eqs. (20), we then obtain the analytical solutions of the displacements in the functionally graded cylindrical panel, i.e.

$$U(R,\theta,Z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[U^{g}_{nm}(R) + U^{p}_{nm}(R)\right]\sin(b\theta)\sin(aZ), \tag{31a}$$

$$V(R,\theta,Z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[V^{g}_{nm}(R) + V^{p}_{nm}(R)\right]\cos(b\theta)\sin(aZ), \tag{31b}$$

$$W(R,\theta,Z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[W^{g}_{nm}(R) + W^{p}_{nm}(R)\right]\sin(b\theta)\cos(aZ). \tag{31c}$$

Moreover, substituting Eqs. (31) into Eqs. (3) and then into Eqs. (2), we then obtain the analytical solutions of the thermo-elastic stress fields in the functionally graded cylindrical panel, i.e.

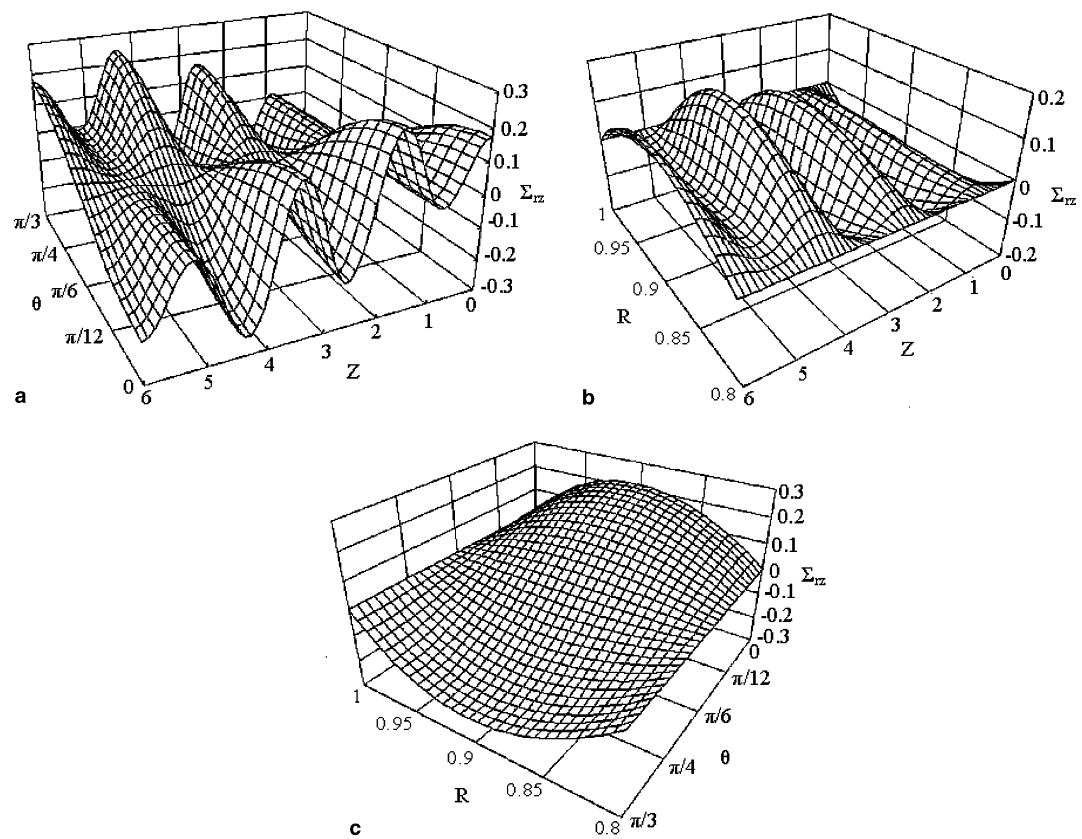

Fig. 7. Distribution of dimensionless shear stress Σ_{rz}: (a) section $R = 0.9$, (b) section $\theta = \pi/4$ and (c) section $Z = 1.5$.

$$\Sigma_r = \frac{E(R)}{(1+\mu)(1-2\mu)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \left[(1-\mu)\frac{\mathrm{d}}{\mathrm{d}R} + \frac{\mu}{R} \right] \left[U_{nm}^{g}(R) + U_{nm}^{p}(R) \right] - \frac{b\mu}{R} \left[V_{nm}^{g}(R) + V_{nm}^{p}(R) \right] + a\mu \left[W_{nm}^{g}(R) + W_{nm}^{p}(R) \right] - (1+\mu)\Omega(R)\Theta_{nm} \right\} \sin(b\theta)\sin(aZ), \tag{32a}$$

$$\Sigma_\theta = \frac{E(R)}{(1+\mu)(1-2\mu)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \left[\mu\frac{\mathrm{d}}{\mathrm{d}R} + \frac{1-\mu}{R} \right] \left[U_{nm}^{g}(R) + U_{nm}^{p}(R) \right] - \frac{(1-\mu)b}{R} \left[V_{nm}^{g}(R) + V_{nm}^{p}(R) \right] + a\mu \left[W_{nm}^{g}(R) + W_{nm}^{p}(R) \right] - (1+\mu)\Omega(R)\Theta_{nm} \right\} \sin(b\theta)\sin(aZ), \tag{32b}$$

$$\Sigma_z = \frac{E(R)}{(1+\mu)(1-2\mu)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \left[\mu\frac{\mathrm{d}}{\mathrm{d}R} + \frac{\mu}{R} \right] \left[U_{nm}^{g}(R) + U_{nm}^{p}(R) \right] - \frac{b\mu}{R} \left[V_{nm}^{g}(R) + V_{nm}^{p}(R) \right] + a(1-\mu) \left[W_{nm}^{g}(R) + W_{nm}^{p}(R) \right] - (1+\mu)\Omega(R)\Theta_{nm} \right\} \sin(b\theta)\sin(aZ), \tag{32c}$$

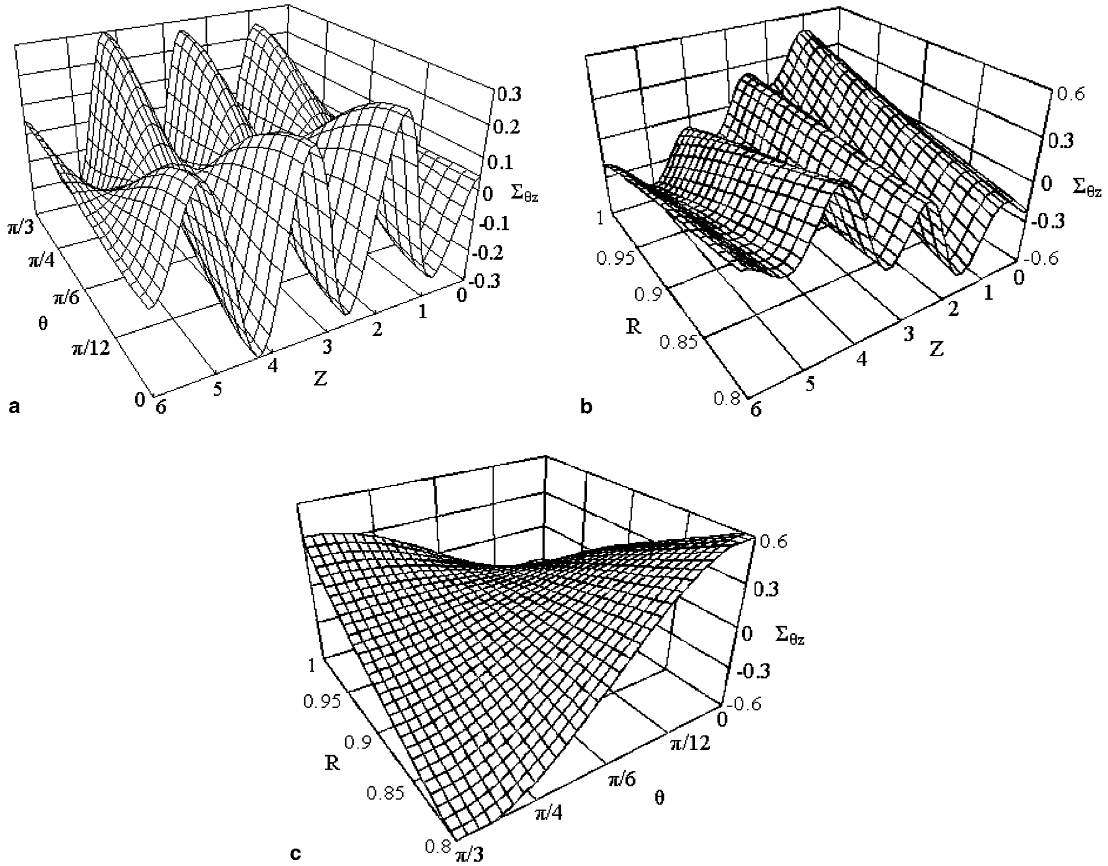

Fig. 8. Distribution of dimensionless shear stress $\Sigma_{\theta z}$: (a) section $R = 0.9$, (b) section $\theta = \pi/4$ and (c) section $Z = 1.5$.

$$\Sigma_{r\theta} = \frac{E(R)}{2(1+\mu)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{b}{R} \left[U^{g}_{nm}(R) + U^{p}_{nm}(R) \right] + \left[\frac{\mathrm{d}}{\mathrm{d}R} - \frac{1}{R} \right] \left[V^{g}_{nm}(R) + V^{p}_{nm}(R) \right] \right\} \cos(b\theta) \sin(aZ), \quad (32\mathrm{d})$$

$$\Sigma_{rz} = \frac{E(R)}{2(1+\mu)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ a \left[U^{g}_{nm}(R) + U^{p}_{nm}(R) \right] + \frac{\mathrm{d}}{\mathrm{d}R} \left[V^{g}_{nm}(R) + V^{p}_{nm}(R) \right] \right\} \sin(b\theta) \cos(aZ), \quad (32\mathrm{e})$$

$$\Sigma_{\theta z} = \frac{E(R)}{2(1+\mu)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ a \left[V^{g}_{nm}(R) + V^{p}_{nm}(R) \right] + \frac{b}{R} \left[V^{g}_{nm}(R) + V^{p}_{nm}(R) \right] \right\} \cos(b\theta) \cos(aZ). \quad (32\mathrm{f})$$

As an example, a mullite/molybdenum functionally graded cylindrical panel with finite length and subjected to nonaxisymmetric thermal and mechanical loads will be considered in what follows, and the three-dimensional solutions of temperature and stress fields will be graphically expressed.

5. Numerical results and discussion

We consider a molybdenum/mullite functionally graded cylindrical panel with the geometric parameters $R_a = 0.8$, $R_b = 1.0$, $L = 6.0$ and $\theta_0 = \pi/3$, as shown in Fig. 1. Outer surface of the panel is pure molybdenum. Inner surface of the panel is the composite of molybdenum/mullite. Both molybdenum and mullite vary continuously from the outer to the inner surfaces of the panel. Such that Young's modulus E, thermal expansion coefficient α and thermal conductivity coefficient λ of the panel vary continuously through the thickness of the panel. E_0, α_0, λ_0 and μ of molybdenum taken from Awaji and Sivakuman (2001) are 330 GPa, $4.9\times10^{-6}\ \mathrm{K}^{-1}$, 138 $\mathrm{W(mK)}^{-1}$ and 0.3, respectively. Without losing the generality, material gradient constants of the molybdenum/mullite functionally graded material are assumed as $m_1 = m_2 = 1.5$ and $m_3 = 2$.

As an example, the following nonaxisymmetric temperature and mechanical loads applied to the inner and outer surfaces of the panel are considered.

$$\Theta_a(\theta, Z) = \frac{1}{4}[1-\cos(12\theta)]\left[1-\cos\left(\frac{2\pi}{3}Z\right)\right] \quad \text{at } R_a = 0.8,$$
$$\Theta_b(\theta, Z) = 0 \quad \text{at } R_b = 1.0,$$
$$Q_a(\theta, Z) = 0.3\sin(3\theta)\sin\left(\frac{\pi}{6}Z\right) \quad \text{at } R_a = 0.8,$$
$$Q_b(\theta, Z) = 0 \quad \text{at } R_b = 1.0.$$

Fig. 2(a)–(c) shows the numerical results of the dimensionless temperature distribution in the functionally graded cylindrical panel. Due to the nonhomogeneity of FGM, the temperature decreases nonlinearly in the radial direction. Near the inner surface of the panel, temperature decreases more quickly than that near the outer surface.

Figs. 3–5 show the numerical results of dimensionless normal stress distribution in the radial, circumferential, and axial directions of the functionally graded cylindrical panel, respectively. One can see the nonuniform distributions of the stresses in the panel. It is seen that the axial stress is larger than the radial stress and is smaller than the circumferential stress. The maximum intensity of the axial stress is about two times of the radial stress and is about one third of the circumferential stress. The radial stress is negative through the whole thickness of the cylindrical panel. The axial and circumferential stresses are negative on the sections near the inner surface and positive on the sections near the outer surface.

Figs. 6–8 show the numerical results of dimensionless shear stress distribution in the functionally graded cylindrical panel, respectively. The shear stress $\Sigma_{r\theta}$ is the largest one among the shear stresses. Clearly, magnitude of the shear stress Σ_{rz} is almost equal to that of the shear stress $\Sigma_{\theta z}$. The maximum intensities of shear stresses Σ_{rz} and $\Sigma_{\theta z}$ are about half of the shear stress $\Sigma_{r\theta}$. Distribution of shear stresses in the axial direction is much more complex than that in the other two directions.

Effects of material gradient constants m_1, m_2 and m_3 on the temperature and stress fields in a functionally graded hollow cylinder have been discussed by Jabbari et al. (2002, 2003). Here, one can assume different values of m_1, m_2 and m_3 to discuss the effects of material gradient constants on the temperature and stress fields in the functionally graded panel considered above. For the sake of brevity, discussions on the effects of m_1, m_2 and m_3 are omitted.

6. Conclusions

Analytical solutions of the three-dimensional temperature and thermo-elastic stress fields in the functionally graded cylindrical panel with finite length are derived in the present paper. The panel is subjected to nonuniform thermal and mechanical loads on the inner and outer surfaces. As an example, the temperature and stress fields in molybdenum/mullite functionally graded cylindrical panel are presented graphically.

Advantage of the present method is its applicability to any material model suggested for functionally graded materials, and the continuous variation of material properties can be included in the solutions. It should be emphasized that the trigonometric series used in this paper are only suitable for the present thermal and mechanical boundary conditions assumed in the paper. For other boundary conditions, one should choose other suitable forms of the trigonometric series.

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